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Yosida frames

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Abstract

A Yosida frame is an algebraic frame in which every compact element is a meet of maximal elements. Yosida frames are used to abstractly characterize the frame of z-ideals of a ring of continuous functions $C(X)$, when X is a compact Hausdorff space. An algebraic frame in which the meet of any two compact elements is compact is Yosida precisely when it is "finitely subfit"; that is, if and only if for each pair of compact elements $a < b$, there is a z (not necessarily compact) such that $a \vee z < 1 = b \vee z$. This is used to prove that if L is an algebraic frame in which the meet of any two compact elements is compact, and L has disjointification and $\dim(L) = 1$, then it is Yosida. It is shown that this result fails with almost any relaxation of the hypotheses. The paper closes with a number of examples, and a characterization of the Bézout domains in which the frame of semiprime ideals is Yosida frame.

MSC: 54B35; 06D22; 06F20

0. Introduction

For some time now, the authors have had a certain fascination with the frame of z-ideals of a ring of continuous functions, and have sought to characterize it abstractly. Theorem 3.8 does just that for $C(X)$, when X is compact Hausdorff. The study of the structure of the prime z-ideals of $C(X)$ over a Tychonoff space X has a long and distinguished history. The reader is referred to [9], and may follow the progression of the more recent, frame-theoretic

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developments, in [10,20,22], and especially the most recent [21], where the Krull dimension of the frame of z-ideals is studied.

This article introduces the concept of a Yosida frame, a structure which is closely related to the frame of z-elements, first introduced in [22]. Apart from the connection to z-elements, Yosida frames appear to be a natural generalization of algebraic regular frames, and there is a wide spectrum of applications of the concept to particular algebraic contexts, several of which are pursued here, in Sections 5 and 6.

The term "Yosida frame" was suggested by the concept of a Yosida space—from the theory of archimedean lattice-ordered groups—which in some sense is its dual. Indeed, many of the examples we have in mind are frames of appropriate subobjects of lattice-ordered groups or rings.

We begin with a basic dictionary of frame-theoretic terms and notation. The knowledgeable reader is probably able to skip most of Section 1, and jump ahead to Section 2.

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1. Frame-theoretic preliminaries

For the background material in this section we refer the reader to [15,17], in most instances, and to [22,20] for additional material on closure operators.

Definition and Remarks 1.1. Throughout this commentary, L is a complete lattice. The top and bottom are denoted 1 and 0, respectively. For $x \in L$, denote the set of elements of L less than or equal to (resp, greater than or equal to) x by $\downarrow x$ (resp. $\uparrow x$).

1. L is *algebraic* if it is generated by its compact elements. Note that $c \in L$ is *compact* if $c \leqslant \bigvee_{i \in I} x_i$ implies that $c \leqslant \bigvee_{i \in F} x_i$ for a suitable finite subset F of I. Then L is algebraic if and only if each $x \in L$ is a supremum of compact elements. $\mathfrak{k}(L)$ stands for the set of compact elements of L. If 1 is compact it is said that L is *compact*.
2. L is said to have the *finite intersection property* (FIP) if for any pair $a, b \in \mathfrak{k}(L)$ it follows that $a \wedge b \in \mathfrak{k}(L)$. Observe that $\mathfrak{k}(L)$ is always closed under taking finite suprema. L is said to be *coherent* if it is compact and has the FIP.
3. L is a *frame* if the following distributive law holds for all $S \subseteq L$:

$$a \wedge \left(\bigvee S\right) = \bigvee \{a \wedge s : s \in S\}.$$

   It is well known that an algebraic lattice is a frame as long as it is distributive.
4. An element $1 > p \in L$ is *prime* if $x \wedge y \leqslant p$ implies that $x \leqslant p$ or $y \leqslant p$. Spec(L) shall denote the set of prime elements of L.
5. Let L be a frame. For each $a \in L$, let $a^{\perp}$ denote the supremum of all $x \in L$ such that $a \wedge x = 0$. Call $p \in L$ a *polar* if it is of the form $p = y^{\perp}$, for some $y \in L$. It is well known that the set $P(L)$ of all polars forms a complete boolean algebra, in which infima agree with those in L.
6. Let L be a frame. Recall that $a \preccurlyeq b$ if $b \vee a^{\perp} = 1$, and that we say that $x \in L$ is *regular* if $x = \bigvee \{a \in L : a \preccurlyeq x\}$. Call L *regular* if each element of L is regular.

7. Let L be a frame and suppose that $j : L \to L$ is a closure operator; jL designates $\{x \in L : j(x) = x\}$; its members are called *j-elements*. Call j a *nucleus* if $j(a \wedge b) = j(a) \wedge j(b)$.
8. A nucleus j on the frame L is *dense* if $j(0) = 0$. Note that j is dense if and only if $0 \in jL$.
9. Suppose that L is an algebraic lattice, and j is a closure operator. Say that j is *inductive* if

$$j(x) = \bigvee\{j(a) : a \in \mathfrak{k}(L), a \leqslant x\}.$$

Then jL is algebraic and $\mathfrak{k}(jL) = j(\mathfrak{k}(L))$. If L is also a frame and j is a nucleus on L, then jL is an algebraic frame as well [22, Section 4]. Observe, in addition, that if L is a frame and j is a nucleus on L, then $\mathrm{Spec}(jL) = \mathrm{Spec}(L) \cap jL$; if, in addition, L is algebraic and j is inductive, then if L has the FIP then so does jL.
10. Suppose that L is an algebraic frame with the FIP and that j is a nucleus on L. Let $\mathrm{Ab}(j)$ stand for the set of all $x \in L$ such that $a \leqslant x$ (with a compact) implies that $j(a) \leqslant x$. Then $\mathrm{Ab}(j)$ is an algebraic frame with the FIP [22, Section 4]. More precisely,

$$\widehat{j}(x) = \bigvee\{j(a) : a \in \mathfrak{k}(L), a \leqslant x\}$$

defines an inductive nucleus such that $\widehat{j}L = \mathrm{Ab}(j)$.
11. Closure operators on L are partially ordered by

$$j_1 \leqslant j_2 \iff \forall x \in L, \quad j_1(x) \leqslant j_2(x) \iff j_2 L \subseteq j_1 L.$$

Under these stipulations $\widehat{j}$ is the largest inductive closure operator below j. It will be convenient to refer to the passage $j \mapsto \widehat{j}$ as the *inductivization* of the nucleus j. Note that j is dense if and only if $\widehat{j}$ is dense.
12. Suppose that L is an algebraic frame. It is said to have *disjointification* if for each pair of compact elements $a, b \in L$, there exist disjoint $c \wedge d = 0$ in $\mathfrak{k}(L)$, such that $c \leqslant a$ and $d \leqslant b$, and $a \vee b = a \vee d = c \vee b$. If L has disjointification then, for each prime $p \in L$, $\uparrow p$ is a chain, and the converse is true as long as L has the FIP. This was first proved by Monteiro; (see [24], or [25, Lemma 2.1], where a proof is given). When $\uparrow p$ is a chain for each $p \in \mathrm{Spec}(L)$, we say that $\mathrm{Spec}(L)$ is a *root system*.

Remarks 1.2. It is worth underscoring that we shall assume and liberally apply Zorn's Lemma, which guarantees that all algebraic frames are *spatial*; that is, each element is a meet of primes.

To conclude this introduction, we record part of [22, Theorem 2.4], on regular algebraic frames. A version of that, without any mention of regularity, appears as [19, Theorem 2.4].

Theorem 1.3. *Let L stand for an algebraic frame. Then L is regular if and only if L has the FIP and each prime of L is maximal.*

2. Yosida frames

Throughout this section L denotes an algebraic frame with the FIP, while $\mathrm{Max}(L)$ stands for the set of all elements $x < 1$ of L and are maximal in this respect; observe that $\mathrm{Max}(L)$ might be empty.

Definition 2.1. L is a *Yosida frame* if and only if every compact element is a meet of maximal elements.

Note that if L is regular, then, by Theorem 1.3, it is necessarily a Yosida frame. As we shall see, it is reasonable to think of a Yosida frame as a generalization of an algebraic regular frame.

The commentary that follows is designed to introduce notation which is handy elsewhere in the article. There is an associated Galois connection, but that is of no consequence here.

For any topological space X, $\mathfrak{O}(X)$ denotes the frame of open sets.

Remarks 2.2. Suppose that $S \subseteq \mathrm{Spec}(L)$; S^* denotes the nucleus defined by $S^*(x) = \bigwedge\{p \in S : p \geqslant x\}$. Put $L_S \equiv \widehat{S^*L}$. We spell it out: $x \in S^*L$ if and only if x is an infimum of primes from S, and $y \in L_S$ precisely when

$$y = \bigvee\{S^*(a) : a \in \mathfrak{k}(L), a \leqslant y\}.$$

For every $x \in L$, let

$$c_S(x) = \{p \in S : x \nleqslant p\}.$$

The sets of the form $c_S(x)$, with $x \in L$, form the open sets of the *hull-kernel* topology on S. Moreover, $S^*(x) = \bigwedge S \backslash c_S(x)$, and $c_S : S^*L \to \mathfrak{O}(S)$ is a frame isomorphism.

In terms of this notation, observe that L is a Yosida frame precisely when $L = L_{\mathrm{Max}}$. For notational economy we shall henceforth abbreviate all occurrences of $\mathrm{Max}(L)^*$ to Max^*.

Here is the example that motivated the term “Yosida frame”.

Example 2.3. Suppose that L is an arbitrary algebraic frame, not necessarily with the FIP. By a routine application of Zorn’s Lemma, and for each $0 < a \in \mathfrak{k}(L)$, the set $c_{\mathrm{Spec}(L)}(a)$ has maximal elements. Indeed, each member of $c_{\mathrm{Spec}(L)}(a)$ lies beneath a maximal element of it. The set of such maximal elements, denoted Y_a, and, with the subspace topology, is called the *Yosida space* of a. The members of Y_a are called the *values* of a. It is easily seen that Y_a is always compact, and also Hausdorff if L has disjointification.

The frame Y_a^*L then consists of all the members of L which are meets of values of a. It is easy to prove that $\mathrm{Max}(L_{Y_a}) = Y_a$.

In Section 3 we use Yosida frames to characterize the frame of z-ideals of a ring $C(X)$ of continuous functions on a compact Hausdorff space. Here is a brief review of the notion of a z-element; in brief, the z-elements of L are those which can be obtained as up-directed joins of members of the class of upper-archimedean elements. We refer the reader to [22, Section 6] for details.

Definition and Remarks 2.4. The algebraic lattice L is an *archimedean lattice* if, for each $c \in \mathfrak{k}(L)$, $\bigwedge \operatorname{Max}(\downarrow c) = 0$. This concept first appeared in [19].

$x \in L$ is *upper-archimedean* if $\uparrow x$ is archimedean. Denote the set of all upper-archimedean elements of L by $\mathbf{a}^{\uparrow}(L)$. It is shown in [22, Lemma 6.2] that $\mathbf{a}^{\uparrow}(L)$ is closed under arbitrary infima. Observe that if L is compact then $\uparrow x$ is compact, and so $x \in \mathbf{a}^{\uparrow}(L)$ precisely when x is an infimum of maximal elements of L. Put $ar(x) = \bigwedge\{z \in \mathbf{a}^{\uparrow}(L) : x \leqslant z\}$; this defines a nucleus, for which $arL = \mathbf{a}^{\uparrow}(L)$. Note that ar is dense if and only if L is archimedean.

Finally, define $z \equiv \widehat{ar}$; that is, $z(x) = \bigvee\{ar(c) : c \leqslant x, c \in \mathfrak{k}(L)\}$, for each $x \in L$.

The following proposition outlines the connection between Yosida frames and the frame of z-elements, and, in particular, gives a characterization of compact Yosida frames.

Proposition 2.5.

(a) *If L is a Yosida frame, then $L = zL$. The converse is true if L is compact.*
(b) *Suppose that L is a Yosida frame. Then* $\operatorname{Max}(L)$ *is dense and L is archimedean.*
(c) *If L is compact, then $zL = L_{\mathrm{Max}}$, and zL is a Yosida frame.*

Proof. (a) follows immediately from the definition of Yosida frames.

(b) Since L is a Yosida frame and 0 is compact it is a meet of maximal elements of L, whence 0 is upper-archimedean and also $\operatorname{Max}(L)$ is dense. Thus, L is archimedean.

(c) As L is compact, the maximal elements of L coincide with the maximal z-elements. This implies that $L_{\mathrm{Max}} \subseteq zL$. On the other hand, if x is upper-archimedean, then, since 1 is compact, x is the meet of maximal elements. Thus, $x \in L_{\mathrm{Max}}$, and it follows that $zL \subseteq L_{\mathrm{Max}}$.

By the same reasoning it is clear that $\operatorname{Max}(zL) = \operatorname{Max}(L)$, whence $L_{\operatorname{Max}(zL)} = zL$; that is, zL is a Yosida frame. □

Here are two question left standing alter Proposition 2.5. We have not been able to answer either one in general.

Question 2.6. *If $L = zL$, then is L necessarily a Yosida frame?*

According to Proposition 2.5(a), yes, if L is compact.

Question 2.7. *Is zL always a Yosida frame?*

Yes, according to Proposition 2.5(c), as long as L is compact. On the other hand, as is shown in Example 5.9, for any Tychonoff space X, the frame of z-ideals of $C(X)$, the ring of all the continuous real-valued functions defined on X, is a Yosida frame. This frame is zL, taking L to be the frame of all convex ℓ-subgroups of $C(X)$, which is not compact, unless X is *pseudocompact*; that is, unless all the continuous real-valued functions defined on X are necessarily bounded.

Notice, incidentally, that the first conclusion in Proposition 2.5(c) is false, in general. That is, while $L_{\mathrm{Max}} \subseteq zL$, they need not coincide (Example 5.8).

3. Coherent normal Yosida frames

The first goal of this section is Theorem 3.5, which spells out how coherent normal Yosida frames arise. A nonnormal coherent Yosida frame is discussed in Example 6.2. The ultimate objective is to characterize the frame of z-ideals of a ring $C(X)$ of continuous functions on a compact Hausdorff space X, and this is achieved with Theorem 3.8.

We begin by recalling the notion of a normal frame.

Definition 3.1. A frame L is *normal* if for each decomposition $1 = a \vee b$, there exist disjoint c and d in L, such that $a \vee d = c \vee b = 1$. If X is a topological space, then $\mathfrak{O}(X)$ is normal if and only if X satisfies the normal separation axiom. We observe, further, that a coherent frame L is normal if and only if every prime is exceeded by a unique maximal element.

Next, we recall an equivalence of categories which is fundamental to the discussion ahead. We refer the reader to [15] for additional details.

Remarks 3.2. (a) If L is a coherent frame then $\mathfrak{k}(L)$ is a distributive lattice with top 1 and bottom 0. Conversely, if B is a distributive lattice with top 1 and bottom 0, then the lattice $\mathscr{I}(B)$ of all ideals of B is a coherent frame. (Recall that $J \subseteq B$ is an *ideal* of B if (i) J is closed under finite suprema and (ii) $0 \leqslant a \leqslant b \in J$ implies that $a \in J$.)

Furthermore, the assignments $B \mapsto \mathscr{I}(B)$ and $L \mapsto \mathfrak{k}(L)$ define the object portion of an equivalence between the category $\mathfrak{D}$ of all distributive lattices with top and bottom, together with all lattice homomorphisms which preserve top and bottom, and the category $\mathfrak{ChFrm}$ of all coherent frames, together with all coherent frame homomorphisms. We note that if $g : B_1 \to B_2$ is a morphism in $\mathfrak{D}$ then $\mathscr{I}(g)$ is defined by

$$\mathscr{I}(g)[\langle T \rangle] = \langle g(T) \rangle.$$

(Note: $\langle T \rangle$ denotes the ideal of B generated by $T \subseteq B$.) Conversely, if $h : L_1 \to L_2$ is a $\mathfrak{ChFrm}$-morphism, then $\mathfrak{k}(h)$ is the restriction to $\mathfrak{k}(L_1)$.

(b) The functor $\mathscr{I}$ may also be regarded as the "free frame" over a distributive lattice with top and bottom. First, denote the function which "embeds" the $\mathfrak{D}$-object B in $\mathscr{I}(B)$ by $\delta_B(a) = \downarrow a$. Now if F is a frame and B a $\mathfrak{D}$-object, and $h : B \to F$ is a morphism in $\mathfrak{D}$, then there is a unique frame morphism $\tilde{h} : \mathscr{I}(B) \to F$ such that $\tilde{h} \cdot \delta_B = h$; i.e., such that the diagram below commutes.

$$\begin{tikzcd}
B \arrow[rr, "\delta_B"] \arrow[dr, "h"'] & & I(B) \arrow[dl, "\tilde{h}"] \\
& F &
\end{tikzcd}$$

In fact $\tilde{h}[\langle T \rangle] = \bigvee h(T)$.

In the sequel B will be a $\mathfrak{D}$-subobject of F which *generates* it, in the sense that each $x \in F$ is a supremum of members of B; in such a situation, h denotes the inclusion. It is an easy exercise to show that $\widetilde{h}$ is then onto F. Likewise, it is easy to prove that if h is one-to-one then $\tilde{h}$ is *dense*, in the sense that if $\widetilde{h}(J) = 0$ then $J = \{0\}$.

We are almost ready for Theorem 3.5. We highlight two preliminaries, as separate lemmas. Using Lemmas 1.3 and 1.6 of [2] we get most of the first lemma. What little remains to be supplied is left to the reader. Note that in the sequel we drop the subscripts in writing $c(x) = c_{\mathrm{Max}(L)}(x)$ (for $x \in L$), which is a typical open set in $\mathrm{Max}(L)$.

Lemma 3.3. *Let L be a compact frame. Then the map c defines a frame isomorphism between* Max^*L *and* $\mathfrak{O}(\mathrm{Max}(L))$. *The following are then equivalent*:

(a) *L is normal*,
(b) $\mathrm{Max}(L)$ *is a normal topological space.*
(c) $\mathrm{Max}(L)$ *is Hausdorff.*
(d) Max^*L *is a compact regular frame.*

The commutative diagram below is at the core of Theorem 3.5. The commutativity is easy, and does not depend on the assumption of normality in the theorem. As the proof is a routine application of basic properties of the hull-kernel topology, it is omitted, save to point out that the restriction of c to $\mathfrak{k}(L)$ is one-to-one on account of the fact that L is Yosida.

Lemma 3.4. *Suppose that L is a coherent Yosida frame. Then the restriction $c : \mathfrak{k}(L) \to$* $\mathfrak{D}(\mathrm{Max}(L))$ *is one-to-one and a $\mathfrak{D}$-morphism. The image $\mathscr{K} = c[\mathfrak{k}(L)]$ generates* $\mathfrak{D}(\mathrm{Max}(L))$, *giving rise to the following commutative diagram*

$$\begin{tikzcd}
\mathfrak{k}(L) \arrow[rr, "i"] \arrow[d, "c"'] & & L \arrow[d, "\mathrm{Max}^*"] \\
\mathfrak{O}(\mathrm{Max}(L)) & & \mathrm{Max}^*L \arrow[ll, "c", "\cong"']
\end{tikzcd}$$

in which i denotes the canonical inclusion of $\mathfrak{k}(L)$ in L.

Theorem 3.5. *Suppose that L is a coherent normal Yosida frame. Then* $\mathrm{Max}(L)$ *is a compact Hansdorff space and $\mathscr{K} = c[\mathfrak{k}(L)]$ is a base for the open sets which is a $\mathfrak{D}$-subobject.*

Conversely, suppose that X is a compact Hausdorff space and that $\mathscr{B}$ is a base for the open sets of X which is a $\mathfrak{D}$-subobject of $\mathfrak{O}(X)$. Then the ideal lattice $\mathscr{I}(\mathscr{B})$ is a coherent normal Yosida frame such that $\mathrm{Max}(\mathscr{I}(\mathscr{B})) \cong X$.

Proof. Let L be a coherent normal Yosida frame. Lemma 3.3 tells us that $\mathrm{Max}(L)$ is compact Hausdorff, and it is easy to see that $\mathscr{K}$ is a base for $\mathfrak{O}(\mathrm{Max}(L))$ and a $\mathfrak{D}$-subobject of it.

Now suppose that X is a compact Hausdorff space, and $\mathscr{B}$ is a generating $\mathfrak{D}$-subobject, of $\mathfrak{O}(X)$. For each point $p \in X$, let

$$\mathfrak{m}_p = \{U \in \mathscr{B} : p \notin U\}.$$

It is a routine exercise to verify that each $\mathfrak{m}_p$ is a maximal ideal of $\mathscr{B}$; the proof uses that X is regular and that $\mathscr{B}$ is a base for the open sets. Now let $\mathfrak{n}$ be any maximal ideal of $\mathscr{B}$. If $\mathfrak{n}$ is distinct from each $\mathfrak{m}_p$, then for each point p there is a member $V_p \in \mathfrak{n}$, containing p, thus furnishing an open cover of X by the V_p. Reducing to a finite subcover, we have $p_1, \ldots, p_k \in X$ such that $X = V_{p_1} \cup \cdots \cup V_{p_k} \in \mathfrak{n}$, which is absurd. This proves that

$$\mathrm{Max}(\mathscr{I}(\mathscr{B})) = \{\mathfrak{m}_p : p \in X\}.$$

It is also easy to show that, for each $V \in \mathscr{B}$, we have $\downarrow V = \bigcap\{\mathfrak{m}_p : p \notin V\}$, which establishes that $\mathscr{I}(\mathscr{B})$ is a Yosida frame.

Finally, the map $x \mapsto \mathfrak{m}_x$ is a homeomorphism of X onto $\mathrm{Max}(\mathscr{I}(\mathscr{B}))$; we leave the verification to the reader. But now we are again in the situation of Lemma 3.3, with the nucleus $\mathrm{Max}^* : \mathscr{I}(\mathscr{B}) \to \mathrm{Max}^*\mathscr{I}(\mathscr{B})$, and the isomorphisms

$$\mathrm{Max}^*\mathscr{I}(\mathscr{B}) \cong \mathfrak{O}(\mathrm{Max}(\mathscr{I}(\mathscr{B}))) \cong \mathfrak{O}(X).$$

From Lemma 3.3 we are able to conclude that, since X is Hausdorff space, $\mathscr{I}(\mathscr{B})$ is a normal frame. This completes the proof. $\square$

Three examples illustrating the theorem deserve mention.

Example 3.6. (a) We suggest that the reader refer to the discussion in 5.1 and Example 5.9 below.

Let X be any Tychonoff space. We consider $z\mathscr{C}(C(X))$, the frame of z-elements of the lattice $\mathscr{C}(C(X))$ of all convex ℓ-subgroups of $C(X)$. To simplify the notation we abbreviate, as in [23], and put $z\mathscr{C}(C(X)) = \mathscr{C}_z(X)$. The members of $\mathscr{C}_z(X)$ are called *z-ideals*. In Example 5.9 it is shown that $\mathscr{C}_z(X)$ is a coherent normal Yosida frame. $\mathrm{Coz}(X)$ denotes the base of all cozerosets of X.

(b) Now assume that X is compact and *zero-dimensional*; this means that the collection $\mathfrak{B}(X)$ of all clopen sets is a base for the open sets of X. Then $\mathscr{I}(\mathfrak{B}(X))$ is canonically isomorphic to $\mathfrak{O}(X)$: in fact, $\tilde{i} : \mathscr{I}(\mathfrak{B}(X)) \to \mathfrak{O}(X)$ is that isomorphism, where i is the inclusion of $\mathfrak{B}(X)$ in $\mathfrak{O}(X)$; (see 3.2(b)). Moreover, $\mathfrak{O}(X)$ is a regular frame, to which the full strength of Theorem 1.3 may be applied.

One may extract this information out of [2, Lemma 1.5]: whenever L is a coherent normal frame, Max^* is an isomorphism precisely when L is regular, which, in turn, happens if and only if $\mathrm{Max}(L)$ is zero-dimensional.

(c) Recall that an open set U is said to be *regular open* if $\mathrm{int}_X \mathrm{cl}_X U = U$. Let $\mathfrak{R}\mathfrak{O}(X)$ denote the collection of all regular open sets of X. Now, in general, $\mathfrak{R}\mathfrak{O}(X)$ is not closed under union; the finite intersection of regular open sets is regular open. It is interesting, nonetheless, to consider bases $\mathscr{B}$ for $\mathfrak{O}(X)$ which are $\mathfrak{D}$-subobjects and happen to consist of regular open sets. (For example, in the so-called almost P-spaces, every cozeroset is regular open.)

Suppose $\mathscr{B}$ is such a base of open sets, and let h denote the inclusion of it in $\mathfrak{O}(X)$. Note that $\mathfrak{RO}(X)$ consists of the polars of $\mathfrak{O}(X)$, and, indeed, $U^{\perp\perp} = \mathrm{int}_X \mathrm{cl}_X U$, for each open set, U. It is then easy to show that for each $U \in \mathscr{B}$, $\downarrow U \in P(\mathscr{I}(\mathscr{B}))$; that is to say, each compact element of $\mathscr{I}(\mathscr{B})$ is a polar. This means that each member of $\mathscr{I}(\mathscr{B})$ is a d-element; that is, an up-directed supremum of polars [22, Section 5].

In preparation for Theorem 3.8, we reconsider $\mathrm{Coz}(X)$, for any Tychonoff space X.

Example 3.7. First, recall the notion of a *σ-frame*: L is a meet-semilattice and countable suprema are also defined, such that the frame law,

$$a \wedge \left(\bigvee S\right) = \bigvee\{a \wedge s : s \in S\},$$

holds for all countable subsets S. The principal example we have in mind is $\mathrm{Coz}(X)$. It is well known—see [9, 1.14]—that $\mathrm{Coz}(X)$ is closed under countable unions and finite intersections. Since the σ-frame law evidently holds here, we have that $\mathrm{Coz}(X)$ is, indeed, a σ-frame.

We briefly consider the frame of all σ-ideals $\mathscr{I}_\sigma(B)$ of a σ-frame B. First, $J \subseteq B$ is a *σ-ideal* if it is nonempty and (i) closed under countable suprema and (ii) $x \in J$ implies that $\downarrow x \subseteq J$. We list the relevant properties of the set $\mathscr{I}_\sigma(B)$ of all σ-ideals of B, leaving it to the reader to supply the proofs:

1. $\mathscr{I}_\sigma(B)$ is a frame under inclusion; the σ-ideal J generated by $\mathscr{S} \subseteq \mathscr{I}_\sigma(B)$ consists of all x which lie beneath a countable supremum $\bigvee_n x_n$, where $x_n \in K_n$ for some $K_n \in \mathscr{S}$. The meet operation in $\mathscr{I}_\sigma(B)$ is set-theoretic intersection.
2. (See [18, Proposition 1.2], of which this item is a special case.) $\mathscr{I}_\sigma(B)$ is the free frame over B, in the following sense: if $g : B \to C$ is any map of σ-frames (that is, g preserves finite infima and countable suprema), into a frame C, then there is a unique frame homomorphism $g^\sigma : \mathscr{I}_\sigma(B) \to C$, such that $g(b) = g^\sigma(\downarrow b)$, for each $b \in B$. Simply define $g^\sigma(\langle b_i : i \in I\rangle_\sigma) = \bigvee_{i\in I} g(b_i)$; ($\langle b_i : i \in I\rangle_\sigma$ denotes the σ-ideal generated by the b_i.)
3. Now let X be a compact Hausdorff space. Then $\mathfrak{O}(X)$ is canonically isomorphic to $\mathscr{I}_\sigma(\mathrm{Coz}(X))$; $\mathfrak{O}(X)$ satisfies the universal Condition 2. Note that in the language of [18, 1.3], $\mathfrak{O}(X)$ is "σ-coherent" and that the cozerosets are the "σ-elements". One may apply [18, Proposition 1.4] to obtain the claim made here.

The reader ought to refer to the diagram in Lemma 3.4 and the labels of the maps in it; the statement of Theorem 3.8 and the subsequent exposition relies on this notation. We formally borrow a definition from [18], in order to facilitate the proof. Suppose that L is a frame; $x \in L$ is said to be a *σ-element* if $x \leqslant \bigvee S$ (for any $S \subseteq L$) implies that $x \leqslant \bigvee T$, for a suitable countable subset T of S. Observe that if $L = \mathfrak{O}(X)$, for a given compact Hausdorff space X, then the open sets which are σ-elements are precisely the cozerosets.

Theorem 3.8. *For any compact Hausdorff spare X, $\mathscr{C}_z(X)$ is a coherent normal Yosida frame, and $c(\mathfrak{k}(\mathscr{C}_z(X))) \cong \mathrm{Coz}(X)$ is a σ-frame consisting of σ-elements of $\mathfrak{O}(\mathrm{Max}(\mathscr{C}_z(X))) \cong \mathfrak{O}(X)$.*

Conversely, suppose that L is a coherent normal Yosida frame. Then, if $c(\mathfrak{k}(L))$ *is a* σ*-frame under the induced operations, and each member of* $\mathrm{Max}^*(i(\mathfrak{k}(L)))$ *is a* σ*-element of* $\mathrm{Max}^*(L)$, *we have the following*:

(a) $\mathfrak{k}(L)$ *is isomorphic to* $\mathrm{Coz}(\mathrm{Max}(L))$, *via the map* c.
(b) $\mathrm{Max}^* L \cong \mathfrak{O}(\mathrm{Max}(L))$, *is the free frame over* $c(\mathfrak{k}(L))$ (*in the sense of* 3.7.2).
(c) $L \cong \mathscr{C}_z(\mathrm{Max}(L))$.

Proof. Regarding the first assertion, the proof of the fact that $\mathscr{C}_z(X)$ is a coherent normal Yosida frame is contained in Example 5.9. Since $\mathfrak{O}(\mathrm{Max}(\mathscr{C}_z(X))) \cong \mathfrak{O}(X)$, and, by [23, Lemma 4.2], the elements of $\mathfrak{k}(\mathscr{C}_z(X))$ correspond to the cozerosets of X, we have that $c(\mathfrak{k}(\mathscr{C}_z(X)))$ consists of σ-elements of $\mathfrak{O}(\mathrm{Max}(\mathscr{C}_z(X)))$.

As to the converse, note that the isomorphism in (b) is part of Theorem 3.5. The claim about freeness follows from the remarks in Example 3.7.

To conclude (a), the reader should observe that, by Lemma 3.4, c maps $\mathfrak{k}(L)$ onto a base of open sets of $\mathrm{Max}(L)$, consisting of cozerosets. Finally, as $c(\mathfrak{k}(L))$ is closed under countable unions, we also have that every cozeroset of $\mathrm{Max}(L)$ lies in $c(\mathfrak{k}(L))$.

For (c), observe that [23, Lemma 4.2] gives that $\mathscr{C}_z(X)$ is canonically frame-isomorphic to $\mathscr{I}(\mathrm{Coz}(X))$, for any Tychonoff space X. □

Finally, we present two corollaries and a remark. We find the first of these corollaries somewhat striking.

Corollary 3.9. *Suppose that L is a coherent normal Yosida frame. Assume that* $c(\mathfrak{k}(L))$ *is a* σ*-frame under the induced operations, and each member of* $\mathrm{Max}^*(i(\mathfrak{k}(L)))$ *is a* σ*-element of* $\mathrm{Max}^*(L)$. *Then L necessarily has disjointification.*

Proof. Apply Theorem 3.8, along with the observations in Example 5.9. □

When the Yosida frame is, in fact, regular, the situation of Theorem 3.8 is even more tightly circumscribed. Recall that a Tychonoff space is called a *P-space* if every cozeroset is clopen. It is well known that every compact P-space is finite. (We refer the reader to the discussion in [9, 4K], and to [9, Theorem 14.29].)

Corollary 3.10. *Suppose that L is coherent and regular and* $\mathfrak{k}(L)$ *is a* σ*-frame under the operations induced from L. Then* $\mathrm{Max}(L)$ *is finite, and* $L = P(L) = zL$ *and isomorphic to the power set of* $\mathrm{Max}(L)$.

Proof. To begin, observe that, since L is regular, $\mathrm{Spec}(L) = \mathrm{Max}(L)$, and $L = \mathrm{Max}^* L$ (Theorem 1.3). Thus, in view of Theorem 3.5, as $\mathfrak{k}(L)$ is closed under countable suprema, so is $c(\mathfrak{k}(L))$. On the other hand, for each compact $a \in L$, $c(a)$ is a clopen set, and since it is compact, it is necessarily a σ-element.

According to Theorem 3.8(a), $c(\mathfrak{k}(L)) = \mathrm{Coz}(\mathrm{Max}(L))$. This means that $\mathrm{Max}(L)$ is a compact P-space, which is perforce finite.

The remaining assertions should now be clear. □

Remarks 3.11. (a) Corollary 3.10 should make it clear how to come up with examples of coherent normal frames with disjointification which fail the conditions of Theorem 3.8. Let L be any coherent regular frame; in view of Theorem 1.3 and the comments in 1.1.12, L necessarily has disjointification. It suffices then to make certain that $\mathrm{Max}(L)$ is infinite; this is enough to insure that $\mathfrak{k}(L)$ is not a σ-frame, and L is not the frame of all z-ideals of a ring of continuous functions over a compact space.

(b) The characterization of $\mathscr{C}_z(X)$ along the lines of Theorem 3.8 when X is not necessarily compact is made more problematic, by virtue of the fact that, without compactness, cozerosets need not be σ-elements. On the other hand, the reader should be aware that if X is a P-space, then every prime ideal of $C(X)$ is minimal—see [9, Theorem 14.29], Thus, $\mathscr{C}_z(X)$ is a coherent regular frame in which the compact elements are closed under countable suprema. If, in addition, the space X itself is Lindelöf, then each member of $\mathfrak{k}(\mathscr{C}_z(X))$ is also a σ-element.

4. Yosida without points

In this section we wish to draw the connection between Yosida frames and the so-called "subfitness" conditions of the literature on frames. The term first appears in [14]; there is different terminology elsewhere, such as in [16].

Definition 4.1. Let L be an arbitrary distributive lattice with top and bottom. It is said to be *subfit* if for each $x < y$ in L there is a $z \in L$ such that $x \vee z < y \vee z = 1$. This extends the usage of the terminology introduced by Isbell in [14].

For algebraic frames we shall require something slightly weaker: we shall say that L is *finitely subfit* if for each $a < b$, both compact, there is a $z \in L$ such that $a \vee z < b \vee z = 1$. If 1 is compact and L is finitely subfit, then it is easy to argue (writing z as a supremum of compact elements) that z may in fact be taken compact as well. Then, in a coherent frame, the finite subfitness reduces to the following: *For each $a < b$, both compact, there is a compact c such that $a \vee c < b \vee c = 1$.* With regard to the equivalence of the categories $\mathfrak{D}$ and $\mathfrak{ChFrm}$, this means that a coherent frame L is Yosida if and only if $\mathfrak{k}(L)$ is subfit.

The connection alluded to above is simple enough. We emphasize, without further comment, that Zorn's Lemma, or else some milder axiom, is needed in the proof.

Proposition 4.2. *Suppose that L is an algebraic frame with the FIP. Then L is a Yosida frame if and only if it is finitely subfit.*

Proof. Suppose that L is a Yosida frame and $a < b$, with both a and b compact. Since a is a meet of maximal elements, there is an $m \in \mathrm{Max}(L)$ such that $a \leqslant m$, but $b \vee m = 1$, and so m witnesses the finite subfitness.

Conversely, suppose L is finitely subfit. Pick $a < b$, both in $\mathfrak{k}(L)$; applying the finite subfitness, we may find an element $z \in L$ such that $y = a \vee z < 1$ and $b \vee z = 1$. Note that in this event we also have $b \vee y = 1$. Since then $b \nleqslant y$, one may use the compactness of b and Zorn's Lemma to conclude, without loss of generality, that y is maximal with respect to the simultaneous conditions $a \leqslant y < 1$ and $b \vee y = 1$. It is then straightforward to check that such a y is, in fact, maximal. This suffices to establish that $a = \mathrm{Max}^*(a)$, for each $a \in \mathfrak{k}(L)$, which means that L is a Yosida frame. $\square$

An interesting consequence of Proposition 4.2, regarding frames of dimension 1, is next. In order to avoid a major digression on the concept of dimension, let us again refer the reader to [20,23] for background, and for our present purposes recall the following information.

Definition and Remarks 4.3. In a frame L, a chain of primes $p_0 < \cdots < p_k$ is said to have *length* k, and that the *dimension* of L, denoted $\dim(L)$, is the supremum of the lengths of chains. More to the point of calculating dimension in an algebraic frame with the FIP and disjointification, the reader is reminded of the "prime-free" criterion for the condition $\dim(L) \leqslant k$, expressed in [20, either Theorem 3.8 or Theorem 4.2]:

$\dim(L) \leqslant k$ *if and only if for each chain* $a_0 < a_1 < \cdots < a_{k+1}$ *of nonzero compact elements of* L *there exist* $b_1, \ldots, b_{k+1} \in \mathfrak{k}(L)$ *such that* $a_i \vee b_{i+1} = a_{i+1}$, *for each* $i = 0, 1, \ldots, k$, *and*

$$a_0 \wedge b_1 \wedge \cdots \wedge b_{k+1} = 0.$$

Say that L is *semisimple* if $\bigwedge \mathrm{Max}(L) = 0$. Note that if L is coherent, then it is semisimple if and only if it is archimedean.

Proposition 4.4. *Suppose that* L *is a semisimple algebraic frame with the FIP and disjointification, and that* $\dim(L) \leqslant 1$. *Then* L *is a Yosida frame*.

Proof. We prove that L is finitely subfit. To that end, assume that $a < b$ are compact elements; if $b = 1$ there is nothing to prove, so we may assume that $b < 1$. By the criterion of 4.3 on dimension, there exist compact elements c and d such that $a \wedge c \wedge d = 0$, $a \vee c = b$ and $b \vee d = 1$. Now, if $a \vee d < 1$, we are done, and therefore we may as well assume that $a \vee d = 1$.

Note that $a \vee (c \wedge d) = b$, so that $c \wedge d$ complements a in $\downarrow b$. Then an easy calculation shows, first, that $c \wedge d = b \wedge d$, and, second, that $d = a^{\perp}$, proving that a is complemented. But then—as the reader will readily verify—$\uparrow a$ too is semisimple, and thus a is the meet of maximal elements. Then we are able to deduce that, since $a < b$, there is a maximal element $m \geqslant a$ such that $b \nleqslant m$, which implies that $b \vee m = 1$. This shows that L is finitely subfit.[1] $\square$

Remarks 4.5. Proposition 4.4 is sharp in several respects:

1. It is false for coherent frames with all of the hypotheses except semisimplicity. Consider the three-element frame, consisting of $0 < a < 1$.

[1] The proof of Proposition 4.4 is probably incorrect. The authors now have a different and simpler proof, which shows as well that the hypothesis of "disjointification" may be dropped entirely. Details will appear elsewhere.

2. It is false for coherent, frames with all of the hypotheses except $\dim(L) \leqslant 1$. The frame in Example 5.7 has dimension 2, but, otherwise, satisfies every hypothesis of the proposition.

A converse to Proposition 4.4 is hopeless; there are coherent Yosida frames of any dimension whatsoever (Example 5.9).

5. Applications in ordered algebraic structures

The material in this section appears separately because some familiarity with ordered algebraic structures is involved. We have provided what we believe is sufficient background information for an enterprising reader, and we trust that the commentaries will illustrate the foregoing frame-theoretic discussion, as well as provide some sense of its scope, a scope which will be amplified further in Section 6.

At this point some basic facts from the theory of lattice-ordered groups are required, along with some of its elementary notation. All groups considered below are abelian.

Definition and Remarks 5.1. For the record, $(G, +, 0, -(\cdot), \vee, \wedge)$ is a *lattice-ordered group* (abbreviated *ℓ-group*) if $(G, +, 0, -(\cdot))$ is a group with $(G, \vee, \wedge)$ as an underlying lattice, and the following distributive laws holds:

$$a + (b \vee c) = (a + b) \vee (a + c).$$

The above distributive law then implies the corresponding distributive law for sum over infimum. The elements of G for which $g \geqslant 0$ are said to be *positive*; the set of positive elements of G is denoted G^+.

We recite the information to be used in this article; in the sequel G stands for an ℓ-group.

1. The underlying lattice of an ℓ-group is distributive [7, Corollary 3.17], and the group structure is torsion free [7, Propositions 3.15 and 3.16].
2. G is *archimedean* if $a, b \in G^+$ and $na \leqslant b$, for each $n \in \mathbb{N}$ imply that $a = 0$.
   We should also introduce the notation $a \ll b$ for positive elements of G: $a \ll b$ means that $na < b$, for each natural number n.
3. A subgroup of G is called an *ℓ-subgroup* if it is a sublattice as well. The ℓ-subgroup C is *convex* if $a \leqslant g \leqslant b$ with $a, b \in C$ implies that $g \in C$. Let $\mathscr{C}(G)$ denote the lattice of all convex ℓ-subgroups of G. $\mathscr{C}(G)$ is a complete sublattice of the lattice of all subgroups of G [7, Theorem 7.5], and an algebraic frame; the latter is due to Birkhoff [7, Proposition 7.10]. $\mathscr{C}(G)$ satisfies the FIP [7, Proposition 7.15], but, in general, fails to be coherent. In $\mathscr{C}(G)$ the convex ℓ-subgroup generated by $a \in G$ is denoted $G(a)$. Each compact element of $\mathscr{C}(G)$ is of this form; this is a restatement of [7, Proposition 7.16].
   Note that G is archimedean if and only if $\mathscr{C}(G)$ is archimedean.
4. The polars of $\mathscr{C}(G)$ are also called polars in this context. We also adopt the conventions that, $a^\perp \equiv G(a)^\perp$, for each $a \in G$; note that $a^{\perp\perp} \equiv G(a)^{\perp\perp}$.

5. It is well known that, for every ℓ-group $G, \mathscr{C}(G)$ is a frame with disjointification. Indeed, if $a, b \geqslant 0$ in G, let $c = a - (a \wedge b)$ and $d = b - (a \wedge b)$; then $G(c)$ and $G(d)$ witness the disjointification of $G(a)$ and $G(b)$.
6. G is said to be *hyperarchimedean* when $\mathscr{C}(G)$ is a regular frame. The topic of hyperarchimedean ℓ-groups was first developed by Conrad in [5]. As we shall presently see, there are plenty of archimedean ℓ-groups for which the frame of convex ℓ-subgroups is Yosida, which are far from being hyperarchimedean.

The basic material on ℓ-groups not covered in 5.1 may be found in [3,7]. We begin this development by describing when $\mathscr{C}(G)$ itself is a Yosida frame. In the formulation that follows, we use the notation of $\mathrm{coz}(f)$ to denote the set of points for which the function f is nonzero, even though the functions in question need not be continuous.

Proposition 5.2. *Let G be an ℓ-group. Then the following are equivalent.*

(a) *$\mathscr{C}(G)$ is a Yosida frame.*

(b) *There is an ℓ-embedding $\varphi : G \to \mathbb{R}^X$ such that, whenever $a, b \geqslant 0$, $b \notin G(a)$ implies that there is an $x \in X$, with $\varphi(a)(x) = 0$ and $\varphi(b)(x) > 0$.*

(c) *There is an ℓ-embedding $\varphi : G \to \mathbb{R}^X$ such that, whenever $a, b \geqslant 0$, $\mathrm{coz}(\varphi(b)) \subseteq \mathrm{coz}(\varphi(a))$ implies that an $m \in \mathbb{N}$ exists, such that $ma \geqslant b$.*

Proof. Observe at the outset that (c) is simply the contrapositive of (b).

Now assume (a). Let $X = \mathrm{Max}(\mathscr{C}(G))$. Note that if $b \notin G(a)$, then, because $\mathscr{C}(G)$ is a Yosida frame, there is a maximal convex ℓ-subgroup M such that $a \in M$, but $b \notin M$. G can be ℓ-embedded in $\mathbb{R}^X$ on account of Hölder's Theorem; in view of the preceding sentence, the representation has the property desired in (b).

That (b) implies (a) follows from the observation that for each $x \in X$, the set $M_x = \{a \in G : \varphi(a)(x) = 0\}$ is either G, or else in $\mathrm{Max}(\mathscr{C}(G))$. $\square$

Recall that an *f-ring* $A = (A, +, 0, -(\cdot), \cdot, \vee, \wedge)$ is a structure which, additively, is an ℓ-group, such that A is a ring, and, in addition, $x \wedge y = 0$ and $a \geqslant 0$ imply that $ax \wedge y = 0$.

For use presently and also in the next section, recall that a ring is *semiprime* if it contains no nonzero nilpotent elements. Here and elsewhere in this article, "ring" means "commutative ring with identity".

To characterize the f-rings for which the frame of convex ℓ-subgroups is Yosida, the following lemma seems essential. Doubtless the result is well known.

Lemma 5.3. *Suppose that A is a semiprime f-ring with identity. Let M be a maximal convex ℓ-subgroup. Then M is a (ring) ideal of A.*

Proposition 5.4. *Any f-ring A for which $\mathscr{C}(A)$ is a Yosida frame is hyperarchimedean.*

Proof. The import of Lemma 5.3 is that any ℓ-embedding $\varphi : A \to \mathbb{R}^X$ satisfying (b) in Proposition 5.2 is a ring homomorphism. For simplicity we identify A with its image under φ.

Now suppose that $0 < f \in A$. Then $\mathrm{coz}(f) = \mathrm{coz}(f^2)$, and so there exist positive integers m and n such that $mf^2 \geqslant f$, and $nf \geqslant f^2$. Thus, for each $x \in \mathrm{coz}(f)$, we have that $1/m \leqslant f(x) \leqslant n$, which means that f is both bounded and bounded away from 0. It is well known that this implies that A is hyperarchimedean [5]. □

The following example indicates that among frames of convex ℓ-subgroups there are natural examples of coherent Yosida frames which are not regular.

Example 5.5. Let G stand for the ℓ-subgroup of $C((0,\infty))$ consisting of all continuous functions which are *piecewise linear*; that is, $f \in G$ if and only if f is continuous and there is a partition of $(0,\infty)$ into finitely many intervals such that the graph of f is a straight line on each one of the intervals. Let $L = \mathscr{C}(G)$.

Observe that the maximal elements of L are of the form

- $M_t = \{f \in G : f(t) = 0\}$, for each $0 > t < \infty$;
- $M_0 = \{f \in G : \lim_{x\to 0} f(x) = 0\}$;
- M_∞, consisting of the bounded functions in G (i.e., the functions which are eventually constant).

We note that $\dim(L) = 1$, and the minimal primes are

- $P_t^+ = \{f \in G : \exists\varepsilon > 0,\ f(x) = 0, \text{ for each } x \in [t, t+\varepsilon)\}$ and
  $P_t^- = \{f \in G : \exists\varepsilon > 0,\ f(x) = 0, \text{ for each } x \in (t-\varepsilon, t]\}$, both for each $0 < t < \infty$;
- $P_0 = \{f \in G : \exists\varepsilon > 0,\ f(x) = 0, \text{ for each } x \in [0,\varepsilon)\}$;
- P_∞, the functions in G which are eventually 0.

With these preliminaries Proposition 5.2(c) is easy to verify, and thus $\mathscr{C}(G)$ is a coherent Yosida frame. (But note that under the representation of the proposition, the constant functions are 0 at infinity.) Moreover, observe the following:

1. G is not hyperarchimedean. Indeed, it can be shown that no nontrivial convex ℓ-subgroup of G is hyperarchimedean.
2. The subring A generated by G—that is, the f-ring of continuous functions on $(0,\infty)$ which are piecewise polynomial—is not hyperarchimedean, and thus $\mathscr{C}(A)$ is not a Yosida frame.

Observe as well that the fact that $\mathscr{C}(G)$ is Yosida can be obtained from Proposition 4.4.

Remarks 5.6. By Proposition 4.4 it can also be shown that for the free abelian ℓ-group on two generators, as well as the free vector lattice on two generators, $\mathscr{C}(G)$ is a Yosida frame. This hinges on [6, Theorem 2.8] ([6, Proposition 3.4] for vector lattices), where it is shown that the dimension of $\mathscr{C}(G)$ is 1 in both of these two cases. (The free abelian ℓ-group on one generator is hyperarchimedean.)

What happens for free objects on a larger basis is unclear, as the dimension of $\mathscr{C}(G)$ is $n-1$ for the free abelian ℓ-group on n generators.

The remaining examples in this section have already been referred to in this article.

Example 5.7. A coherent archimedean frame of dimension 2, which is not a Yosida frame. (See 4.5.2.)

Once again, we define an ℓ-group G and consider $\mathscr{C}(G)$. G is the group of real sequences described below:

$$G = \{f \in \mathbb{R}^{\mathbb{N}} : \exists m \in \mathbb{N}, \exists a, b, c \in \mathbb{R}, \forall n \geqslant m,\ f(n) = an^2 + bn + c\}.$$

Loosely speaking, G consists of all real sequences which are eventually real polynomial of degree $\leqslant 2$. G is lattice-ordered coordinatewise. Then $L = \mathscr{C}(G)$ is compact and archimedean (5.1.3).

Next, we describe the members of $\mathrm{Spec}(L)$. Let

- $M_k = \{f \in G : f(k) = 0\}$, for each positive integer k;
- P be the subgroup consisting of all eventually zero sequences;
- Q be the subgroup of all eventually constant sequences;
- M be the subgroup of all sequences which are eventually linear.

Note that $P \subset Q \subset M$. The reader may easily check that

$$\mathrm{Spec}(L) = \{M_k : k \in \mathbb{N}\} \cup \{P, Q, M\},$$

with Q being the only prime which is neither maximal nor minimal.

In fact, note that $\mathrm{Max}(L) = \{M_k : k \in \mathbb{N}\} \cup \{M\}$. Let 1 and i denote the constant sequence and the identity function, respectively. Then $G(1) \subseteq Q$, whereas $G(i) \subseteq \mathrm{Max}^*(G(1)) = M$, but $G(i) \nsubseteq Q$. Thus, $Q \notin L_{\mathrm{Max}}$.

Thus, we have that $\dim(L) = 2$. Furthermore, as $zL \neq L$ (Proposition 2.5(c)), L is not Yosida, as promised.

To document the next example we give the appropriate references as we go. We shall use the fact that for any archimedean f-ring A with identity, $\mathscr{C}_z(A)$ consists of ring ideals, whence it is compact. This follows from [13, Proposition 3.1], and a discussion of the details may be found in [21, 8.2(b)].

Example 5.8. An archimedean frame L with the FIP and disjointification, such that zL is coherent, yet in which $\mathrm{Max}(L)$ is empty.

Let X be a compact, extremally disconnected space without P-points; (see [8]). Let $D(X)$ be the ring of all functions (with pointwise operations) which are continuously defined on X and have values in the extended real numbers, such that $f^{-1}(\mathbb{R})$ is dense, for each $f \in D(X)$. The absence of P-points ensures that $L = \mathscr{C}(D(X))$ has no maximal elements; this is proved in [4], but otherwise seems to be folklore. As has already been observed, zL is compact.

For the final example we refer the reader to [9, Chapter 2] for a discussion of z-ideals, and to [12, Theorem 3.3] for a crucial observation linking the notion of a z-ideal (in the context of rings of continuous functions) to the frame-theoretic concept of z-element.

Example 5.9. For each Tychonoff space X, $\mathscr{C}_z(X)$ is a coherent normal Yosida frame.

$C(X)$ denotes the ring of all continuous real-valued functions defined on X; the operations are pointwise, making $C(X)$ a semiprime ring. By defining the lattice operations pointwise as well, $C(X)$ is made into an ℓ-group (and, indeed, an f-ring, although this does not matter here).

Recall the complementary notions of a zeroset and a cozeroset: they are the sets of the form

$$Z(f) = \{x \in X : f(x) = 0\} \quad \text{and} \quad \operatorname{coz}(f) = \{x \in X : f(x) \neq 0\}, \quad f \in C(X).$$

To say that X is a *Tychonoff space* is to say that X is Hausdorff and the cozerosets of X form a base for $\mathfrak{O}(X)$.

According to the conventions of [9, Chapter 2], a subgroup J of $C(X)$ is a *z-ideal* if $f \in J$ and $\operatorname{coz}(g) \subseteq \operatorname{coz}(f)$ imply that $g \in J$. Note that every z-ideal is an ideal of the ring $C(X)$ and also semiprime. It is shown in [12, Theorem 3.3] that the z-ideals of $C(X)$ are precisely the elements of $\mathscr{C}_z(X)$.

Note that $\mathscr{C}_z(X)$ has disjointification (and is therefore normal): this follows because z is an inductive nucleus on an algebraic frame, namely $\mathscr{C}(C(X))$, which has disjointification.

Next, observe the following. The compact elements of $\mathscr{C}_z(X)$ are the ideals of the form

$$\langle f\rangle_z \equiv \{g \in C(X) \colon \operatorname{coz}(g) \subseteq \operatorname{coz}(f)\}.$$

Moreover, each $\langle f\rangle_z$ is clearly the intersection of all the maximal ideals of the form $M_p = \{h \in C(X) : h(p) = 0\}$, where p ranges over $Z(f)$. Each M_p is, obviously, a z-ideal and maximal in $\mathscr{C}_z(X)$. Thus, $\mathscr{C}_z(X)$ is a Yosida frame, as asserted. Further, $\langle 1\rangle_z = C(X)$, signifying that $\mathscr{C}_z(X)$ is compact and, therefore, coherent.

Finally, according to [23, Theorem 5.3], for compact X, $\mathscr{C}_z(X)$ has finite dimension if and only if X is scattered of finite Cantor–Bendixson index. Recall, at the same time, that $\mathscr{C}_z(X)$ is always a coherent Yosida frame with disjointification.

6. An application to integral domains

In this section we illustrate with an example from classical commutative algebra. We remind the reader that all rings considered in this article are commutative and possess an identity.

It is well known that the lattice of all semiprime ideals, $\operatorname{Rad}(A)$, of a ring A is a coherent algebraic frame. Indeed, by a theorem of Hochster [11], every coherent, algebraic frame arises in this manner. To recall, $\mathfrak{r}$ is *semiprime*, if $x^2 \in \mathfrak{r}$ implies that $x \in \mathfrak{r}$. For each semiprime ring A, $\operatorname{Spec}(\operatorname{Rad}(A))$ is none other than the classical spectrum $\operatorname{Spec}(A)$; i.e., the set of all prime ideals. For each $a \in A$, let

$$j_{\mathrm{Rad}}(a) = \{r \in A : \exists n \in \mathbb{N},\ x \in A,\ \text{with } r^n = xa\}.$$

Then it is easily seen that $j_{\mathrm{Rad}}(a)$ is the least semiprime ideal containing a. In addition, observe that since

$$j_{\mathrm{Rad}}(a) \cap j_{\mathrm{Rad}}(b) = j_{\mathrm{Rad}}(ab), \quad a, b \in A,$$

$\operatorname{Rad}(A)$ satisfies the FIP.

We set up the specifics in the commentary which follows.

Remarks 6.1. Let A stand for a semiprime commutative ring with identity. For brevity, we write $\mathrm{Max}(A)$ for $\mathrm{Max}(\mathrm{Rad}(A))$, and $\mathrm{Max}^*(A)$ in place of $\mathrm{Max}^*(\mathrm{Rad}(A))$.

We consider the *semisimple* rings; that is, the rings having trivial Jacobson radical, or, equivalently, the rings in which the intersection of all the maximal ideals is zero. Obviously, $\mathrm{Max}(A)$ is dense if and only if A is semisimple. Observe as well that since $\mathrm{Rad}(A)$ is a coherent frame, $\mathrm{Rad}(A)$ is archimedean precisely when A is semisimple. Moreover, the frame $z\mathrm{Rad}(A)$ is the set of directed unions of intersections of maximal ideals (Proposition 2.5(c)), and a Yosida frame.

Finally, $\mathrm{Rad}(A)$ itself is a Yosida frame if and only if every principal semiprime ideal of A is an intersection of maximal ideals. Putting it differently, $\mathrm{Rad}(A)$ is a Yosida frame if and only if, for each $a, b \in A$, whenever a fails to divide every power b^n (for $n \in \mathbb{N}$), then there is a homomorphism $\varphi : A \to K$, for a suitable field K, such that $\varphi(b) \neq 0$, while $\varphi(a) = 0$.

It is not hard to see that the ring $\mathbb{Z}$ of integers, and, indeed, every principal ideal domain, has the above property.

We amplify the above comment about $\mathbb{Z}$.

Example 6.2. A coherent Yosida frame of dimension 1 which is not normal.

Let $L = \mathrm{Rad}(\mathbb{Z})$. It is well known that $\mathrm{Max}(\mathbb{Z})$ is the set of all nonzero prime ideals of $\mathbb{Z}$. Since $\mathbb{Z}$ is a principal ideal domain, this means that each element of L is compact and a meet of maximal elements. However, L is not normal, because $\mathrm{Max}(\mathbb{Z})$ is homeomorphic to $\mathbb{N}$ with the finite complement topology, which is not regular. (This example turns up as [22, 3.5(ii)], albeit from a slightly different point of view.) That $\dim(L) = 1$ is well known.

If A is a Bézout domain, then we are able to characterize when $\mathrm{Rad}(A)$ is a Yosida frame in terms of the group of divisibility *G(A)* of A. Our chief reference for features of the group of divisibility of a domain is [1, Chapter 11].

Definition and Remarks 6.3. Throughout this item A stands for an integral domain.

(a) Recall that A is a *Bézout* domain if each finitely generated ideal of A is principal. This is easily seen to be equivalent to the condition: for each $a, b \in A$ there is a $d \in A$ (unique up to multiplicative units) such that d divides a and b and $d = ua + vb$, for suitable $u, v \in A$.

(b) To each domain one associates a group, namely $G(A) \equiv qA^\#/U(A)$, where qA is the field of fractions of A and *U(A)* is the group of multiplicative units of A; $qA^\# = qA\backslash\{0\}$. $G(A)$ admits a natural partial ordering, defined by $xU(A) \leqslant yU(A)$ if and only if $yx^{-1} \in A$. If A is a Bézout domain then this partial ordering defines a lattice structure, making $G(A)$ into a lattice-ordered group [1, Theorem 11.1].

The canonical homomorphism w then also satisfies the inequality

$$w(x + y) \geqslant w(x) \wedge w(y), \quad \text{provided } x + y \neq 0.$$

The convention in most of the literature is to immediately turn $G(A)$ into an additive group, via $xU(A) + yU(A) = xyU(A)$, after which the homomorphic law reads logarithmically:

$w(xy) = w(x) + w(y)$. w is the *valuation* associated with A. $G(A)$ is the *group of divisibility of* A.

(c) The following are well known;

1. $A = \{x \in qA : w(x) \geqslant 0\} \cup \{0\}$ and $U(A) = \{x \in qA : w(x) = 0\}$.
2. There is a one-to-one, order inverting isomorphism between $\mathrm{Spec}(A)$ and $\mathrm{Spec}(\mathscr{C}(G(A)))$, given by

$$\mathfrak{p} \mapsto \overline{\mathfrak{p}} \equiv \langle \{w(x) : x \notin \mathfrak{p}\} \rangle_\ell,$$

where $\langle S \rangle_\ell$ stands for the convex ℓ-subgroup generated by S. (See [1, Theorem 11.3], which goes further.) Evidently then, under this correspondence, the maximal ideals of A correspond to the minimal primes of $\mathscr{C}(G(A))$.

We now have the following straightforward lemma.

Lemma 6.4. *Let A be a Bézout domain. Then $b \in A$ lies in the semiprime ideal generated by $a \in A$ if and only if $w(a) \in G(w(b))$.*

Proof. $b^n = ra$, for suitable $r \in A$ and positive integer n, precisely when $nw(b) \geqslant w(a)$; that is, if and only if $w(a) \in G(w(b))$. $\square$

We then obtain the following characterization of Yosida frames.

Theorem 6.5. *Let A be a Bézout domain.* $\mathrm{Rad}(A)$ *is a Yosida frame if and only if each principal convex ℓ-subgroup of $G(A)$ is an intersection of minimal primes of $\mathscr{C}(G(A))$; if and only if for each $g \in G(A)$, $G(g)$ is a polar.*

Proof. Assume that $\mathrm{Rad}(A)$ is a Yosida frame. It clearly suffices to prove that (in $G(A)$) if $w(a) \notin G(w(b))$, there is a minimal prime convex ℓ-subgroup of $G(A)$ containing $w(b)$ but not $w(a)$. By Lemma 6.4 we have that a fails to divide every power b^n, and therefore that a maximal ideal $\mathfrak{m}$ of A exists containing a but not b. Applying the correspondence of 6.3(c).2, we have a minimal element in $\mathrm{Spec}(\mathscr{C}(G(A)))$, $\overline{\mathfrak{m}}$, such that $w(b) \in \overline{\mathfrak{m}}$, but $w(a) \notin \overline{\mathfrak{m}}$, as promised.

The reverse direction is then obvious; the final claim is as well. $\square$

A special note ought to be made of the case of a valuation domain. Recall that A is a *valuation domain* if its lattice of ideals is totally ordered under inclusion. It is easily seen that any valuation domain is Bézout, and that A is a valuation domain if and only if $G(A)$ is totally ordered.

Corollary 6.6. *Let A be a valuation domain. Then* $\mathrm{Rad}(A)$ *is a Yosida frame if and only if $G(A)$ is archimedean.*

Proof. This is an immediate consequence of Theorem 6.5 and Hölder's Theorem [7, Theorem 24.16 and Corollary 24.17] $\square$

References

[1] M. Anderson, T. Feil, Lattice-ordered Groups; an Introduction, Reidel Texts in Mathematical Sciences, D. Reidel Publishers Co., Kluwer Academic Publisher, Dordrecht, 1988.
[2] B. Banaschewski, Pointfree topology and the spectrum of f-rings, in: W.C. Holland, J. Martinez (Eds.), Ordered Algebraic Structures, Kluwer Academic Publishers, Dordrecht, 1997, pp. 123–148.
[3] A. Bigard, K. Keimel, S. Wolfenstein, Groupes et Anneaux Réticulés, Lecture Notes, vol. 608, Springer, Berlin, Heidelberg, New York, 1977.
[4] D. Chambless, Tulane Dissertation, 1971, New Orleans.
[5] P.F. Conrad, Epi-archimedean groups, Czechoslovak Math. J. 24 (1974) 1–27.
[6] M.R. Darnel, The structure of free abelian ℓ-groups and free vector lattices, Algebra Universalis 23 (1986) 99–110.
[7] M.R. Darnel, The Theory of Lattice-Ordered Groups, Pure & Applied Mathematics, vol. 187, Marcel Dekker, Basel, Hong Kong, New York, 1995.
[8] R. Engelking, General Topology, Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989.
[9] L. Gillman, M. Jerison, Rings of Continuous Functions, Graduate Texts in Mathematics, vol. 43, Springer, Berlin, Heidelberg, New York, 1976.
[10] M. Henriksen, J. Martínez, R.G. Woods, Spaces X in which all prime z-ideals of $C(X)$ are either minimal or maximal, Comm. Math. Univ. Carol. 44 (2) (2003) 261–294.
[11] M. Hochster, Prime ideal structure in commutative rings, Trans. Amer. Math. Soc. 142 (1969) 749–758.
[12] C.B. Huijsmans, B. de Pagter, On z-ideals and d-ideals in Riesz spaces, J. Indag. Math. 42 (1980) 183–195.
[13] C.B. Huijsmans, de Pagter, Ideal theory in f-algebras, Trans. Amer. Math. Soc. 269 (1982) 225–245.
[14] J.R. Isbell, Atomless parts of spaces, Math. Scand. 31 (1972) 5–32.
[15] P.T. Johnstone, Stone Spaces, Cambridge Studies in Advanced Mathematics, vol. 3, Cambridge University Press, London, 1982.
[16] P.T. Johnstone, Wallman compactifications of locales, Houston J. Math. 10 (2) (1984) 201–206.
[17] A. Joyal, M. Tierney, An extension of the Galois theory of Grothendieck, Memoirs AMS, vol. 309, 1984, AMS, Providence.
[18] J.J. Madden, κ-frames, J. Pure Appl. Algebra 70 (1991) 107–127.
[19] J. Martínez, Archimedean lattices. Algebra Universalis 3 (1973) 247–260.
[20] J. Martínez, Dimension in algebraic frames, Czechoslovak Math. J., to appear.
[21] J. Martínez, Unit and kernel systems in algebraic frames, submitted.
[22] J. Martínez, E.R. Zenk, When an algebraic frame is regular, Algebra Universalis 50 (2003) 231–257.
[23] T. Martínez, E.R. Zenk, Dimension in algebraic frames, II: applications to frames of ideals in $C(X)$, submitted.
[24] A. Monteiro, L'Arithmétique des filtres et les espaces topologiques, Segundo Symposium de Matemática; Villavicencio (Mendoza) (1954) 129–162.
[25] J.T. Snodgrass, C. Tsinakis, Finite-valued algebraic lattices, Algebra Universalis 30 (1993) 311–318.